

SUPERCONNECTIONS AND AFFINE MANIFOLDS

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ABSTRACT. By using the Mathai-Quillen superconnection construction of the Thom class, we show that the Euler characteristic of a compact affine manifold equals to zero. This confirms an old conjecture of S. S. Chern.

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0. INTRODUCTION

By the celebrated Gauss-Bonnet-Chern theorem [4] and its ramifications, if the tangent bundle TM of a closed smooth manifold M admits a flat connection preserving certain Riemannian metric on TM , then the Euler characteristic of M vanishes: $\chi(M) = 0$. It is natural to ask what would happen if one drops the metric preserving condition.

Benzécri [1] proved that if a closed surface admits a torsion free flat connection, then its Euler characteristic vanishes. Milnor [11] refined this result by dropping the torsion free condition. However, examples due to Smillie [13] show that this torsion free condition can not be dropped in higher dimensions.

Recall that a manifold admitting a torsion free flat connection is an affine manifold, that is, having a covering by coordinate charts such that the coordinate transformation in overlapping charts is linear.

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The remaining question, widely called the Chern conjecture, is whether the Euler characteristic of a closed affine manifold vanishes (cf. [5, Problem V] and [6, pp. 727]). For complete affine manifolds, a positive answer to this question was given by Kostant and Sullivan [8]. For more recent advances, see Bucher-Gelander [3] and Klingler [7].

In this paper, we confirm the Chern conjecture in its full generality.

Theorem 0.1. *The Euler characteristic of a closed affine manifold equals to zero.*

The usual difficulty concerning this problem is the lack of the metric preserving property for the underlying flat connection. Our simple observation is that if one applies the Chern-Weil theory to an exterior algebra version of the Mathai-Quillen formalism on the geometric construction of the Thom class [10], one can resolve this difficulty. On the other hand, a weighted Riemannian metric constructed from affine coordinate charts (cf. (2.13)), as well as the ideas of transgressions (which go back to [4]), also play essential roles in obtaining the required vanishing result.

This paper is organized as follows. In Section 1, we present the exterior algebra version of the Mathai-Quillen formalism. In Section 2, we prove Theorem 0.1.

1. EXTERIOR ALGEBRA VERSION OF THE MATHAI-QUILLEN FORMALISM

In this section, we present an exterior algebra version of the Mathai-Quillen formalism [10]. That is, we replace the spinor bundle considered in [10] by the exterior algebra bundle. This gives a formula for the Euler characteristic for an arbitrary connection.

Let $\pi : TM \rightarrow M$ be the tangent vector bundle of a $2n$ dimensional closed manifold M . Let ∇^{TM} be a connection on TM . Then it induces a connection $\nabla^{\Lambda^*(T^*M)}$ on $\Lambda^*(T^*M)$, which preserves the \mathbf{Z}_2 -splitting $\Lambda^*(T^*M) = \Lambda^{\text{even}}(T^*M) \oplus \Lambda^{\text{odd}}(T^*M)$.

Let g^{TM} be a Euclidean metric on TM . For any $Z \in TM$, let $Z^* \in T^*M$ be its metric dual. As usual (cf. [14, Section 4.3]), let $c(Z)$ be the Clifford action on $\Lambda^*(T^*M)$ defined by

$$(1.1) \quad c(Z) = Z^* \wedge -i_Z.$$

Then

$$(1.2) \quad c(Z)^2 = -|Z|_{g^{TM}}^2.$$

For any $T > 0$, let A_T be the superconnection [12] on $\pi^*\Lambda^*(T^*M)$ defined by

$$(1.3) \quad A_T = \pi^*\nabla^{\Lambda^*(T^*M)} + Tc(Z),$$

where $c(Z)$ now acts on $(\pi^*\Lambda^*(T^*M))|_Z$.

By (1.2) and (1.3), $\exp(A_T^2)$ is of exponential decay along vertical directions of TM .

Theorem 1.1. *The following identity holds,*

$$(1.4) \quad \chi(M) = \left(\frac{1}{2\pi} \right)^{2n} \int_{TM} \text{tr}_s [\exp(A_T^2)].$$

Proof. By the Chern-Weil theory for superconnections (cf. [2, Proposition 1.43]) and the above mentioned exponential decay property of $\exp(A_T^2)$, we see that the right hand side of (1.4) does not depend on the choice of ∇^{TM} and $T > 0$. Thus, we may well assume that $T = 1$ and that ∇^{TM} preserves g^{TM} . Then one can follow the strategy in [10].

In fact, since the computation is local, one may well assume that TM is spin. Then one has the following decomposition in terms of the spinor bundle $S(TM) = S_+(TM) \oplus S_-(TM)$,

$$(1.5) \quad \Lambda^*(T^*M) = (S_+(TM) \oplus S_-(TM)) \hat{\otimes} (S_+^*(TM) \oplus S_-^*(TM)),$$

and $c(Z)$ now acts on $(\pi^*S(TM))|_Z$. Moreover, $\nabla^{\Lambda^*(T^*M)}$ decomposes to $\nabla^{S(TM)} \otimes \text{Id}_{S^*(TM)} + \text{Id}_{S(TM)} \otimes \nabla^{S^*(TM)}$.

From (1.3) and (1.5), one gets

$$(1.6) \quad \text{tr}_s [\exp(A_1^2)] = \text{tr}_s \left[\exp \left((\pi^* \nabla^{S(TM)} + c(Z))^2 \right) \right] \cdot \text{tr}_s \left[\exp \left((\pi^* \nabla^{S^*(TM)})^2 \right) \right].$$

By [10, Theorem 4.5], one has

$$(1.7) \quad \left(\frac{\sqrt{-1}}{2\pi} \right)^n \text{tr}_s \left[\exp \left((\pi^* \nabla^{S(TM)} + c(Z))^2 \right) \right] = (-1)^n \det \left(\frac{\sinh(\pi^* R^{TM}/2)}{\pi^* R^{TM}/2} \right)^{\frac{1}{2}} U,$$

where $R^{TM} = (\nabla^{TM})^2$ and U is the Thom form constructed in [10, (4.7)].

By [9, Proposition III.11.24], one has that for any spin Euclidean vector bundle E of even rank, with corresponding spinor bundle $S(E) = S_+(E) \oplus S_-(E)$,

$$(1.8) \quad \text{ch}(S_+^*(E) - S_-^*(E)) = \frac{e(E)}{\hat{A}(E)},$$

where $e(E)$ is the Euler class of E , and $\hat{A}(E)$ is the Hirzebruch \hat{A} class of E .

By (1.6)-(1.8) and [10, Theorem 4.10], which integrates U along vertical fibers, one has,

$$(1.9) \quad \left(\frac{1}{2\pi} \right)^{2n} \int_{TM} \text{tr}_s [\exp(A_1^2)] = \left\langle \frac{e(TM)}{\hat{A}(TM)^2}, [M] \right\rangle.$$

Since $e(TM)$ is already of top degree, we get (1.4) from (1.9). \square

2. AFFINE MANIFOLDS AND THE CHERN CONJECTURE

This section is organized as follows. In Section 2.1, we present the basic setting of the affine manifold under consideration. In Sections 2.2-2.4, we prove Theorem 0.1.

2.1. Affine structures. Recall that a smooth manifold M is called an affine manifold if there exists a local coordinate system covering $\{(U_\alpha, (x_\alpha^i))\}$ of M such that for any α, β with $U_\alpha \cap U_\beta \neq \emptyset$, one has

$$(2.1) \quad x_\beta = x_\alpha A_{\alpha\beta} + B_{\alpha\beta},$$

where $x_\alpha = (x_\alpha^1, \dots, x_\alpha^{\dim M})$, while $(A_{\alpha\beta}), (B_{\alpha\beta})$ are two constant matrices of orders $(\dim M) \times (\dim M)$ and $1 \times (\dim M)$ over $U_\alpha \cap U_\beta$ respectively. Without loss of generality (by passing to a double covering if necessary), we assume that

$$(2.2) \quad \det(A_{\alpha\beta}) > 0$$

for any α, β with $U_\alpha \cap U_\beta \neq \emptyset$ and choose compatible orientations on U_α 's. Then M is an oriented affine manifold carrying the induced orientation.

From now on, we assume that M is a closed oriented affine manifold of dimension $2n$. Let $\{(U_\alpha, (x_\alpha^i))\}$, $\alpha = 1, \dots, N$, be an affine structure on M .¹ Then it induces an affine structure $\{(\pi^{-1}(U_\alpha), (x_\alpha^i, y_\alpha^i))\}$ on $\pi : TM \rightarrow M$.

We have on the total manifold TM that, when restricted to $\pi^{-1}(U_\alpha \cap U_\beta)$,

$$(2.3) \quad x_\beta = x_\alpha A_{\alpha\beta} + B_{\alpha\beta}, \quad y_\beta = y_\alpha A_{\alpha\beta}, \quad A_{\alpha\beta} = \left(\frac{\partial x_\beta^j}{\partial x_\alpha^i} \right),$$

$$(2.4) \quad \frac{\partial}{\partial x_\alpha^i} = \sum_j \frac{\partial x_\beta^j}{\partial x_\alpha^i} \frac{\partial}{\partial x_\beta^j}, \quad \frac{\partial}{\partial y_\alpha^i} = \sum_j \frac{\partial x_\beta^j}{\partial x_\alpha^i} \frac{\partial}{\partial y_\beta^j},$$

$$(2.5) \quad dx_\beta^j = \sum_i \frac{\partial x_\beta^j}{\partial x_\alpha^i} dx_\alpha^i, \quad dy_\beta^j = \sum_i \frac{\partial x_\beta^j}{\partial x_\alpha^i} dy_\alpha^i.$$

By (2.4), $T(TM)$ carries a canonical splitting $T(TM) = T^H(TM) \oplus T^V(TM)$, where $T^H(TM)$ (resp. $T^V(TM)$) is spanned by $\{\frac{\partial}{\partial x_\alpha^k}\}$'s (resp. $\{\frac{\partial}{\partial y_\alpha^k}\}$'s). Also, by (2.5), one has

Lemma 2.1. *The dual bundle $(T^V M)^*$ is spanned by $\{dy_\alpha^k\}$'s.*

Without loss of generality, we assume that each U_α is of the form $U_\alpha = \{x_\alpha : \sum_i (x_\alpha^i)^2 < 1\}$. Let $h : [0, 1] \rightarrow [0, 1]$ be a non-increasing smooth function such that $h(t) = 1$ near $t = 0$ and $h(t) = \exp(-1/(1-t)^2)$ near $t = 1$.

For any U_α , let r_α be the radius function on U_α defined by $r_\alpha(x_\alpha) = (\sum_i (x_\alpha^i)^2)^{1/2}$. Let ρ_α be the function on U_α defined by

$$(2.6) \quad \rho_\alpha(x_\alpha) = h(r_\alpha).$$

¹The ordering of U_α 's, while arbitrary, will play an important role in what follows.

Then ρ_α extends to a smooth function on M such that $\text{Supp}(\rho_\alpha) \subseteq \overline{U}_\alpha$ and

$$(2.7) \quad \rho_\alpha > 0 \quad \text{over } U_\alpha.$$

For any function ρ on M , we use the same notation ρ to denote its lift $\pi^*\rho$.

Remark 2.2. Indeed, (2.6) need only to hold near the boundaries ∂U_α 's.

Without loss of generality, we also assume that all the boundaries ∂U_α 's intersect to each other complete transversally. Then any point on M lies on at most $2n$ different boundaries. Moreover, the set

$$(2.8) \quad \mathbf{B} = \{p \in M : p \text{ lies on } 2n \text{ different boundaries}\}$$

consists of finite points.

For any $p \in \mathbf{B}$, set $U_p = \cap_{p \in U_\alpha} U_\alpha$. Then there exists a (sufficiently small) open neighborhood W_p of $p \in M$ with $\overline{W}_p \subset U_p$ such that for any different $p, q \in \mathbf{B}$, one has $\overline{W}_p \cap \overline{W}_q = \emptyset$.

Remark 2.3. For any $p \in \mathbf{B}$, let $U_{\alpha_1}, \dots, U_{\alpha_{N_p}}$, with $\alpha_1 > \dots > \alpha_{N_p}$, be the open coordinate charts containing p . In view of Remark 2.2, we assume that $\rho_{\alpha_1}, \dots, \rho_{\alpha_{N_p}}$ equal to 1 on W_p . Also, we denote by $U_{\beta_1}, \dots, U_{\beta_{2n}}$, with $\beta_1 > \dots > \beta_{2n}$, the open coordinate charts such that $\partial U_{\beta_1}, \dots, \partial U_{\beta_{2n}}$ intersect at p . Moreover, by making W_p small enough, we may well assume that $\overline{W}_p \cap \overline{U}_\alpha = \emptyset$ for any $\alpha \notin \{\alpha_1, \dots, \alpha_{N_p}, \beta_1, \dots, \beta_{2n}\}$.

As a final notation, we set

$$(2.9) \quad \mathbf{B}_+ = \{p \in \mathbf{B} : \beta_{2n} > \alpha \text{ if } p \in U_\alpha\}.$$

2.2. Superconnections and the Chern conjecture. Clearly, the affine structure defines canonically a flat connection on $\pi^*\Lambda^*(T^*M)$ by that on each $\pi^{-1}(U_\alpha)$,

$$(2.10) \quad \pi^*\nabla^{\Lambda^*(T^*M)} = \sum_k dx_\alpha^k \otimes \frac{\partial}{\partial x_\alpha^k} + \sum_k dy_\alpha^k \otimes \frac{\partial}{\partial y_\alpha^k}.$$

Define on $\pi^{-1}(U_\alpha)$ that

$$(2.11) \quad Y = \sum_k y_\alpha^k \frac{\partial}{\partial x_\alpha^k} \in \Gamma(\pi^*(TM)).$$

By (2.3) and (2.4), Y is a well-defined canonical section of $\pi^*(TM)$.

In what follows, for clarity, we will decorate elements in $\pi^*\Lambda^*(T^*M)$, which is considered as a vector bundle, with a $\widehat{}$ notation.

For any $T > 0$, let $\widehat{\eta}_T \in \Gamma(\pi^*(T^*M))$ be defined by

$$(2.12) \quad \widehat{\eta}_T = \sum_{\alpha=1}^N \rho_\alpha T^\alpha \sum_k y_\alpha^k \widehat{dx}_\alpha^k.$$

Remark 2.4. For any $T > 0$, if we give TM the metric defined by

$$(2.13) \quad g_T^{TM} = \sum_{\alpha=1}^N \rho_\alpha T^\alpha \sum_k (dx_\alpha^k)^2,$$

then $\widehat{\eta}_T$ is the metric dual of \widehat{Y} (with respect to $\pi^* g_T^{TM}$).

Set as in (1.1) that

$$(2.14) \quad c_T(\widehat{Y}) = \widehat{\eta}_T \wedge -i_{\widehat{Y}}.$$

Then $c_T(\widehat{Y})$ is an odd endomorphism of $(\pi^* \Lambda^*(T^*M))|_Y$. Moreover, one has

$$(2.15) \quad |Y|_{g_T^{TM}}^2 = -c_T(\widehat{Y})^2 = \sum_{\alpha,k} \rho_\alpha T^\alpha (y_\alpha^k)^2.$$

For any $T > 0$, let A_T be the superconnection on $\pi^* \Lambda^*(T^*M)$ defined by

$$(2.16) \quad A_T = \pi^* \nabla^{\Lambda^*(T^*M)} + c_T(\widehat{Y}).$$

By Theorem 1.1, one has

$$(2.17) \quad \chi(M) = \left(\frac{1}{2\pi} \right)^{2n} \int_{TM} \text{tr}_s [\exp(A_T^2)].$$

We need to compute $\int_{TM} \text{tr}_s [\exp(A_T^2)]$, which does not depend on $T > 0$.

From (2.10)-(2.12) and (2.14)-(2.16), one has

$$(2.18) \quad \begin{aligned} A_T^2 &= \left[\pi^* \nabla^{\Lambda^*(T^*M)}, \sum_{\alpha,k} \rho_\alpha T^\alpha y_\alpha^k \widehat{dx}_\alpha^k - i_{\widehat{Y}} \right] - |Y|_{g_T^{TM}}^2 \\ &= \sum_{\alpha,k} T^\alpha \left(d\rho_\alpha y_\alpha^k \widehat{dx}_\alpha^k + \rho_\alpha dy_\alpha^k \widehat{dx}_\alpha^k \right) - \sum_k dy_\alpha^k \otimes i_{\widehat{\frac{\partial}{\partial x_\alpha^k}}} - |Y|_{g_T^{TM}}^2. \end{aligned}$$

Set on each $\pi^{-1}(U_\alpha)$ that

$$(2.19) \quad d^H = \sum_k dx_\alpha^k \frac{\partial}{\partial x_\alpha^k}, \quad d^V = \sum_k dy_\alpha^k \frac{\partial}{\partial y_\alpha^k}, \quad \widehat{d}^V = \sum_k \widehat{dx}_\alpha^k \frac{\partial}{\partial y_\alpha^k}.$$

By (2.4) and (2.5), the sums do not depend on α . Thus d^H , d^V and \widehat{d}^V are well-defined over TM .

From (2.15), (2.18) and (2.19), one has

$$(2.20) \quad A_T^2 = \frac{1}{2} d^H \widehat{d}^V \left(|Y|_{g_T^{TM}}^2 \right) + \sum_{\alpha,k} T^\alpha \rho_\alpha dy_\alpha^k \widehat{dx}_\alpha^k - \sum_k dy_\alpha^k \otimes i_{\widehat{\frac{\partial}{\partial x_\alpha^k}}} - |Y|_{g_T^{TM}}^2.$$

Set

$$(2.21) \quad B_T^2 = \frac{1}{2} d^H \widehat{d}^V \left(|Y|_{g_T^{TM}}^2 \right) - \sum_k dy_\alpha^k \otimes i \widehat{\frac{\partial}{\partial x_\alpha^k}} - |Y|_{g_T^{TM}}^2.$$

By Lemma 2.1, (2.20), (2.21) and a simple degree counting along vertical directions, one sees that

$$(2.22) \quad \mathrm{tr}_s [\exp (A_T^2)] = \mathrm{tr}_s [\exp (B_T^2)].$$

By (2.19) and (2.21), we see that if we exchange \widehat{dx}_α^k and dy_α^k , we get the same supertrace of $\exp(B_T^2)$. Thus, we have

$$(2.23) \quad \begin{aligned} \mathrm{tr}_s [\exp (B_T^2)] &= \exp \left(\frac{1}{2} d^H d^V |Y|_{g_T^{TM}}^2 - |Y|_{g_T^{TM}}^2 \right) \mathrm{tr}_s \left[\exp \left(- \sum_k \widehat{dx}_\alpha^k i \widehat{\frac{\partial}{\partial x_\alpha^k}} \right) \right] \\ &= \exp \left(\frac{1}{2} d^H d^V |Y|_{g_T^{TM}}^2 - |Y|_{g_T^{TM}}^2 \right). \end{aligned}$$

Proposition 2.5. *There exists a suitable choice of ρ_α 's verifying the conditions in Remark 2.3 such that for any $p \in M \setminus \mathbf{B}_+$, there is an open neighborhood V_p of p in M such that for any non-negative smooth function $f \in C^\infty(M)$ supported in V_p , one has*

$$(2.24) \quad \lim_{T \rightarrow +\infty} \int_{TM} f \exp \left(\frac{1}{2} d^H d^V |Y|_{g_T^{TM}}^2 - |Y|_{g_T^{TM}}^2 \right) = 0.$$

Moreover, (2.24) still holds for $p \in \mathbf{B}_+$ if one allows $f = 1$ near p .

Theorem 0.1 can be obtained from Proposition 2.5 as follows.

We choose a finite selection of V_p 's in Proposition 2.5 so that they form an open covering of M . Let $\{f_{V_p}\}$'s be a partition of unity subordinate to this open covering. We assume that each $p \in \mathbf{B}_+$ is covered by only one V_p on which $f_{V_p} = 1$ near p . Since \mathbf{B}_+ consists of finite points, the existence of such a covering and partition of unity is clear.

By (2.23) and Proposition 2.5, one gets

$$(2.25) \quad \lim_{T \rightarrow +\infty} \int_{TM} \mathrm{tr}_s [\exp (B_T^2)] = \sum \lim_{T \rightarrow +\infty} \int_{TM} f_{V_p} \exp \left(\frac{1}{2} d^H d^V |Y|_{g_T^{TM}}^2 - |Y|_{g_T^{TM}}^2 \right) = 0.$$

From (2.17), (2.22) and (2.25), we get $\chi(M) = 0$.

The proof of Theorem 0.1 is completed.

The proof of Proposition 2.5 will be carried out in the next two subsections.

2.3. Proof of Proposition 2.5: the case of $p \notin \mathbf{B}_+$. We assume in this subsection that $p \notin \mathbf{B}_+$. By (2.15) and (2.19), one has near p that

$$(2.26) \quad \frac{1}{2} d^H d^V |Y|_{g_T^{TM}}^2 = \sum_{\alpha, k} d\rho_\alpha T^\alpha y_\alpha^k dy_\alpha^k.$$

Thus, one has

$$(2.27) \quad \exp \left(\frac{1}{2} d^H d^V |Y|_{g_T^{TM}}^2 - |Y|_{g_T^{TM}}^2 \right) = \prod_\alpha \left(1 + d\rho_\alpha T^\alpha \sum_k y_\alpha^k dy_\alpha^k \right) \exp \left(-|Y|_{g_T^{TM}}^2 \right).$$

For simplicity, we denote

$$(2.28) \quad h_\alpha = \sum_k (y_\alpha^k)^2.$$

Take $p \in M$. We assume that among $\{U_\alpha\}_{\alpha=1}^N$, there are exactly N_p elements U_{α_i} , with $\alpha_1 > \dots > \alpha_{N_p}$, containing p .

If p does not lie on any boundary of U_α 's, then from (2.27) and (2.28), one has

$$(2.29) \quad \left\{ \exp \left(\frac{1}{2} d^H d^V |Y|_{g_T^{TM}}^2 - |Y|_{g_T^{TM}}^2 \right) \right\}^{(4n)} = \sum_{\{\alpha_{i_j}\}} \prod_{j=1}^{2n} \left(d\rho_{\alpha_{i_j}} T^{\alpha_{i_j}} \frac{dh_{\alpha_{i_j}}}{2} \right) \exp \left(-|Y|_{g_T^{TM}}^2 \right)$$

near $\pi^{-1}(p)$, where α_{i_j} runs through α_i , $1 \leq i \leq N_p$. Each α_{i_j} appears at most once in a product. Moreover, by (2.15), one has near $\pi^{-1}(p)$ that

$$(2.30) \quad |Y|_{g_T^{TM}}^2 \geq \frac{1}{2} \rho_{\alpha_1}(p) T^{\alpha_1} \sum_k (y_{\alpha_1}^k)^2.$$

Since $\rho_{\alpha_1}(p) > 0$, from (2.30), one sees that there is an open neighborhood V_p of $p \in M$ such that for any $f \in C^\infty(M)$ supported in V_p , when $T \gg 0$,

$$(2.31) \quad \left| \int_{TM} f \prod_{j=1}^{2n} \left(d\rho_{\alpha_{i_j}} T^{\alpha_{i_j}} dh_{\alpha_{i_j}} \right) \exp \left(-|Y|_{g_T^{TM}}^2 \right) \right| = O \left(\prod_{j=1}^{2n} T^{\alpha_{i_j} - \alpha_1} \right) = O \left(\frac{1}{T^{n(2n-1)}} \right).$$

Formula (2.24) follows from (2.29) and (2.31).

We now assume that p lies on the boundaries of some U_α 's. To be more precise, we assume that p lies on the boundaries of $\{U_{\beta_i}\}_{i=1}^{M_p}$ with $\beta_1 > \beta_2 > \dots > \beta_{M_p}$. Then by (2.27), the terms we need to consider, near $\pi^{-1}(p)$, are of the form

$$(2.32) \quad \prod_{h=1}^k \left(d\rho_{\beta_{i_h}} T^{\beta_{i_h}} dh_{\beta_{i_h}} \right) \prod_{j=1}^{2n-k} \left(d\rho_{\alpha_{i_j}} T^{\alpha_{i_j}} dh_{\alpha_{i_j}} \right) \exp \left(-|Y|_{g_T^{TM}}^2 \right).$$

For simplicity, we assume that $\beta_{i_1} > \dots > \beta_{i_k}$ and $\alpha_{i_1} > \dots > \alpha_{i_{2n-k}}$.

By (2.15), there exists constant $c_1 > 0$ such that one has, near $\pi^{-1}(p)$,

$$(2.33) \quad |Y|_{g_T^{TM}}^2 \geq c_1 \left(\sum_{j=1}^k \rho_{\beta_{i_j}} T^{\beta_{i_j}} + \rho_{\alpha_1}(p) T^{\alpha_1} \right) \sum_l (y_{\alpha_1}^l)^2.$$

From (2.33), one sees that there exists constant $C_1 > 0$ such that when $T \gg 0$, the following formula holds near $p \in M$,

$$(2.34) \quad \left| \int_{TM/M} \prod_{h=1}^k \left(d\rho_{\beta_{i_h}} T^{\beta_{i_h}} dh_{\beta_{i_h}} \right) \prod_{j=1}^{2n-k} \left(d\rho_{\alpha_{i_j}} T^{\alpha_{i_j}} dh_{\alpha_{i_j}} \right) \exp \left(-|Y|_{g_T^{TM}}^2 \right) \right| \\ \leq C_1 \left| \frac{\prod_{h=1}^k \left(d\rho_{\beta_{i_h}} T^{\beta_{i_h}} \right) \prod_{j=1}^{2n-k} \left(d\rho_{\alpha_{i_j}} T^{\alpha_{i_j}} \right)}{\left(\sum_{j=1}^k \rho_{\beta_{i_j}} T^{\beta_{i_j}} + \rho_{\alpha_1}(p) T^{\alpha_1} \right)^{2n}} \right|,$$

where for a form $Fd\text{vol}$, we use the notation $|Fd\text{vol}| = |F|d\text{vol}$, and $|Fd\text{vol}| \leq |Gd\text{vol}|$ means $|F| \leq |G|$.

Recall that the boundaries of $U_{\beta_{i_h}}$'s intersect transversally to each other.

Since the function h used in the definition of ρ_{α} 's in (2.6) is non-increasing, one has

$$(2.35) \quad \left| \frac{\prod_{h=1}^k \left(d\rho_{\beta_{i_h}} T^{\beta_{i_h}} \right) \prod_{j=1}^{2n-k} \left(d\rho_{\alpha_{i_j}} T^{\alpha_{i_j}} \right)}{\left(\sum_{j=1}^k \rho_{\beta_{i_j}} T^{\beta_{i_j}} + \rho_{\alpha_1}(p) T^{\alpha_1} \right)^{2n}} \right| \leq \left| \prod_{h=1}^k \frac{d\rho_{\beta_{i_h}} T^{\beta_{i_h}}}{\rho_{\beta_{i_h}} T^{\beta_{i_h}} + \rho_{\alpha_1}(p) T^{\alpha_1}} \cdot \prod_{j=1}^{2n-k} \frac{d\rho_{\alpha_{i_j}} T^{\alpha_{i_j}}}{\rho_{\alpha_1}(p) T^{\alpha_1}} \right| \\ = \left| \prod_{h=1}^k \left(d \log \left(\rho_{\beta_{i_h}} + \rho_{\alpha_1}(p) T^{\alpha_1 - \beta_{i_h}} \right) \right) \cdot \prod_{j=1}^{2n-k} \frac{d\rho_{\alpha_{i_j}} T^{\alpha_{i_j}}}{\rho_{\alpha_1}(p) T^{\alpha_1}} \right|.$$

It is easy to see that for $a > 0$ sufficiently small and $T \gg 0$, the integration (for each $1 \leq h \leq k$) along $1 - a \leq r_{\beta_{i_h}} \leq 1$ of $d \log(\rho_{\beta_{i_h}} + \rho_{\alpha_1}(p) T^{\alpha_1 - \beta_{i_h}})$ is of $O(\log T)$ (resp. $O(\frac{1}{T})$) if $\beta_{i_h} > \alpha_1$ (resp. $\beta_{i_h} < \alpha_1$). Also, if $k \leq 2n - 2$, then one has that, in view of (2.31), when $T \geq 1$,

$$(2.36) \quad \prod_{j=1}^{2n-k} \frac{T^{\alpha_{i_j}}}{T^{\alpha_1}} \leq \frac{1}{T}.$$

From (2.34)-(2.36), one finds that if $\alpha_1 > \beta_{i_k}$ or if $k \leq 2n - 2$, then there exists a sufficiently small open neighborhood V_p of $p \in M$ such that for any smooth function $f \in C^\infty(M)$ supported in V_p ,

$$(2.37) \quad \lim_{T \rightarrow +\infty} \int_{TM} f \prod_{h=1}^k \left(d\rho_{\beta_{i_h}} T^{\beta_{i_h}} dh_{\beta_{i_h}} \right) \prod_{j=1}^{2n-k} \left(d\rho_{\alpha_{i_j}} T^{\alpha_{i_j}} dh_{\alpha_{i_j}} \right) \exp \left(-|Y|_{g_T^{TM}}^2 \right) = 0.$$

We need only to consider the case of $k = M_p = 2n - 1$ with $\beta_{2n-1} > \alpha_1$, and the case of $M_p = 2n$.

For the case of $k = M_p = 2n - 1$ and $\beta_{2n-1} > \alpha_1$ in (2.32), if $\alpha_{i_1} < \alpha_1$, then by (2.35) one still gets (2.37). Thus, we need only to deal with the term

$$\begin{aligned}
 (2.38) \quad & \prod_{h=1}^{2n-1} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) (d\rho_{\alpha_1} T^{\alpha_1} dh_{\alpha_1}) \exp \left(-|Y|_{g_T^{TM}}^2 \right) \\
 &= d^V \left(\prod_{h=1}^{2n-1} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \rho_{\alpha_1}^{-1} d\rho_{\alpha_1} \exp \left(-|Y|_{g_T^{TM}}^2 \right) \right) \\
 &\quad - \prod_{h=1}^{2n-1} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \rho_{\alpha_1}^{-1} d\rho_{\alpha_1} \left(\sum_{i=2}^{N_p} \rho_{\alpha_i} T^{\alpha_i} dh_{\alpha_i} \right) \exp \left(-|Y|_{g_T^{TM}}^2 \right),
 \end{aligned}$$

from which and the Stokes formula, we still get (2.37) via (2.35).

For the case of $M_p = 2n$, by Remark 2.3 one has $\rho_{\alpha_j} = 1$ ($j = 1, \dots, N_p$) near $p \in M$. Thus one may assume $k = 2n$. Since $\beta_{2n}(p) < \alpha_1(p)$, (2.37) follows from (2.35).

Thus (2.37), which implies (2.24), holds for $p \notin \mathbf{B}_+$.

2.4. Proof of Proposition 2.5: the case of $p \in \mathbf{B}_+$. We now suppose $p \in \mathbf{B}_+$. Recall that p is an intersection point of $\{\partial U_{\beta_i}\}_{i=1}^{2n}$ with $\beta_1 > \dots > \beta_{2n}$, and that p lies in the open coordinate charts $\{U_{\alpha_j}\}_{j=1}^{N_p}$ with $\alpha_1 > \dots > \alpha_{N_p}$. Moreover, $\beta_{2n} > \alpha_1$. Also recall that by Remark 2.3, one has that $\rho_{\alpha_j} = 1$ ($1 \leq j \leq N_p$) on W_p .

Let $0 \leq \phi \leq 1$ be a smooth function on M such that $\text{Supp}(\phi) \subset W_p$ and that ϕ equals to 1 near each $p \in M$. Moreover, we may and we will assume that $\text{Supp}(\phi) \subset M$ is a smooth manifold with boundary, with the boundary $\partial(\text{Supp}(\phi))$ intersecting to ∂U_{β_i} 's complete transversally. The existence of ϕ is clear.

We need only to prove the following identity,

$$(2.39) \quad \lim_{T \rightarrow +\infty} \int_{TM} \phi \prod_{h=1}^{2n} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \exp \left(-|Y|_{g_T^{TM}}^2 \right) = 0,$$

where

$$(2.40) \quad |Y|_{g_T^{TM}}^2 = \sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i} + \sum_{i=1}^{N_p} T^{\alpha_i} h_{\alpha_i}.$$

For brevity, we set the notation on $\pi^{-1}(W_p)$ that

$$(2.41) \quad |Y|_T^2 = \sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i} + T^{\alpha_1} h_{\alpha_1}.$$

Lemma 2.6. *The following identity holds,*

$$(2.42) \quad \lim_{T \rightarrow +\infty} \int_{TM} \phi \prod_{h=1}^{2n} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \exp \left(-|Y|_{g_{TM}}^2 \right) \\ = \lim_{T \rightarrow +\infty} \int_{TM} \phi \prod_{h=1}^{2n} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \exp \left(-|Y|_T^2 \right).$$

Proof. Set on $\pi^{-1}(W_p)$ that $i_Y = \sum_k y_\alpha^k i_{\frac{\partial}{\partial y_\alpha^k}}$, which does not depend on α . Then one has

$$(2.43) \quad (d^H - i_Y)^2 = 0.$$

For any $1 \leq \alpha \leq N$, one verifies that

$$(2.44) \quad (d^H - i_Y) (\rho_\alpha dh_\alpha) = d\rho_\alpha dh_\alpha - 2\rho_\alpha h_\alpha.$$

From (2.40), (2.41), (2.43) and (2.44), one deduces that

$$(2.45) \quad \frac{\phi}{2^{2n}} \prod_{h=1}^{2n} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \exp \left(-|Y|_{g_{TM}}^2 \right) - \frac{\phi}{2^{2n}} \prod_{h=1}^{2n} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \exp \left(-|Y|_T^2 \right) \\ = \left\{ \phi \prod_{i=1}^{2n} \left(e^{(d^H - i_Y)(\rho_{\beta_i} T^{\beta_i} dh_{\beta_i}/2)} \right) e^{-T^{\alpha_1} h_{\alpha_1}} \left(e^{(d^H - i_Y) \sum_{i=2}^{N_p} T^{\alpha_i} dh_{\alpha_i}/2} - 1 \right) \right\}^{(4n)} \\ = \left\{ (d^H - i_Y) \left(\phi \prod_{i=1}^{2n} \left(e^{(d^H - i_Y)(\rho_{\beta_i} T^{\beta_i} dh_{\beta_i}/2)} \right) e^{-T^{\alpha_1} h_{\alpha_1}} \left(\sum_{i=2}^{N_p} T^{\alpha_i} dh_{\alpha_i}/2 \right) \frac{e^{-\sum_{i=2}^{N_p} T^{\alpha_i} h_{\alpha_i}} - 1}{-\sum_{i=2}^{N_p} T^{\alpha_i} h_{\alpha_i}} \right) \right. \\ \left. - d\phi \prod_{i=1}^{2n} \left(e^{(d^H - i_Y)(\rho_{\beta_i} T^{\beta_i} dh_{\beta_i}/2)} \right) e^{-T^{\alpha_1} h_{\alpha_1}} \left(\sum_{i=2}^{N_p} T^{\alpha_i} dh_{\alpha_i}/2 \right) \frac{e^{-\sum_{i=2}^{N_p} T^{\alpha_i} h_{\alpha_i}} - 1}{-\sum_{i=2}^{N_p} T^{\alpha_i} h_{\alpha_i}} \right\}^{(4n)}.$$

From (2.41), (2.44), (2.45) and the Stokes formula, one gets

$$(2.46) \quad \int_{TM} \phi \prod_{h=1}^{2n} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \exp \left(-|Y|_{g_{TM}}^2 \right) - \int_{TM} \phi \prod_{h=1}^{2n} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \exp \left(-|Y|_T^2 \right) \\ = \int_{TM} d\phi \left(\sum_{h=1}^{2n} \prod_{i \neq h} (d\rho_{\beta_i} T^{\beta_i} dh_{\beta_i}) \right) \left(\sum_{i=2}^{N_p} T^{\alpha_i} dh_{\alpha_i} \right) \frac{e^{-\sum_{i=2}^{N_p} T^{\alpha_i} h_{\alpha_i}} - 1}{\sum_{i=2}^{N_p} T^{\alpha_i} h_{\alpha_i}} \exp \left(-|Y|_T^2 \right).$$

Recall that $\alpha_1 > \alpha_i$ for $i \geq 2$. Also, for any $t \geq 0$, one has

$$(2.47) \quad 0 \leq \frac{1 - e^{-t}}{t} \leq 1.$$

By (2.41), (2.46), (2.47) and proceeding as in (2.35), one gets (2.42). \square

Since for any $1 \leq i \leq 2n$, $U_{\beta_i} \cap U_{\alpha_1} \neq \emptyset$, one may extend h_{β_i} to $\pi^{-1}(W_p)$. It is clear that there exist $c > c' > 0$ such that for any $1 \leq i \leq 2n-1$, one has $ch_{\beta_{2n}} \geq h_{\beta_i} \geq c'h_{\beta_{2n}}$. It also holds if we replace h_{β_i} by h_{α_1} .

For any $R > 0$, set $D^V(R) = \{(x, y) \in TM : x \in \text{Supp}(\phi), h_{\beta_{2n}}(y) \leq R\}$ and $\partial D^V(R) = \{(x, y) \in D^V(R) : h_{\beta_{2n}}(y) = R\}$.

Lemma 2.7. *For any $x \in \text{Supp}(\phi)$ and $R > 0$, the following identity holds,*

$$(2.48) \quad \int_{D_x^V(R)} \prod_{h=1}^{2n} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \exp \left(- \sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i} \right) = 0.$$

Proof. If $\rho_{\beta_1}(x) > 0$, then one has, by the Stokes formula,

$$(2.49) \quad \begin{aligned} & \int_{D_x^V(R)} \prod_{h=1}^{2n} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \exp \left(- \sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i} \right) \\ &= \int_{D_x^V(R)} d^V \left(\frac{d\rho_{\beta_1}}{\rho_{\beta_1}} \prod_{h=2}^{2n} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \exp \left(- \sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i} \right) \right) \\ &= \int_{\partial D_x^V(R)} \frac{d\rho_{\beta_1}}{\rho_{\beta_1}} \prod_{h=2}^{2n} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \exp \left(- \sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i} \right) = 0, \end{aligned}$$

as $h_{\beta_{2n}} = R$ is constant on $\partial D_x^V(R)$. If $\rho_{\beta_1}(x) = 0$, then (2.48) holds trivially. \square

From (2.41), (2.43) and (2.44), one deduces that

$$(2.50) \quad \begin{aligned} & \prod_{h=1}^{2n} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \exp(-|Y|_T^2) - \prod_{h=1}^{2n} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \exp \left(- \sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i} \right) \\ &= \prod_{h=1}^{2n} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) e^{-\sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} (\exp(-T^{\alpha_1} h_{\alpha_1}) - 1) \\ &= 2^{2n} \left\{ \prod_{h=1}^{2n} \left(e^{(d^H - i_Y)(\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}/2)} \right) \left(e^{(d^H - i_Y)(T^{\alpha_1} dh_{\alpha_1}/2)} - 1 \right) \right\}^{(4n)} \\ &= 2^{2n} \left\{ (d^H - i_Y) \left(\prod_{h=1}^{2n} \left(e^{(d^H - i_Y)(\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}/2)} \right) dh_{\alpha_1} \frac{e^{-T^{\alpha_1} h_{\alpha_1}} - 1}{-2h_{\alpha_1}} \right) \right\}^{(4n)}. \end{aligned}$$

From (2.48), (2.50) and the Stokes formula, one gets for any $R > 0$ that

$$\begin{aligned}
 (2.51) \quad & \int_{D^V(R)} \phi \prod_{h=1}^{2n} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \exp(-|Y|_T^2) \\
 &= -2^{2n} \int_{D^V(R)} d\phi \prod_{h=1}^{2n} \left(e^{(d^H - i_Y)(\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}/2)} \right) dh_{\alpha_1} \frac{1 - e^{-T^{\alpha_1} h_{\alpha_1}}}{2 h_{\alpha_1}} \\
 &= - \int_{D^V(R)} d\phi \left(\sum_{i=1}^{2n} \prod_{h \neq i} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \right) e^{-\sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} dh_{\alpha_1} \frac{1 - e^{-T^{\alpha_1} h_{\alpha_1}}}{h_{\alpha_1}}.
 \end{aligned}$$

Let $g(x)$ be the function defined by

$$(2.52) \quad g(x) = \int_0^x \frac{1 - e^{-t}}{t} dt.$$

Then one has

$$(2.53) \quad dh_{\alpha_1} \frac{1 - e^{-T^{\alpha_1} h_{\alpha_1}}}{h_{\alpha_1}} = T^{\alpha_1} dh_{\alpha_1} \frac{1 - e^{-T^{\alpha_1} h_{\alpha_1}}}{T^{\alpha_1} h_{\alpha_1}} = dg(T^{\alpha_1} h_{\alpha_1}).$$

For any $x \in \text{Supp}(d\phi)$, one has $x \neq p$. If one of $\rho_{\beta_i}(x) > 0$, then by (2.47) and (2.53), we see that by proceeding as in (2.35), there exists a sufficiently small open neighborhood V_x of $x \in M$ such that for any $f \in C^\infty(M)$ supported in V_x , one has

$$(2.54) \quad \lim_{T \rightarrow +\infty} \lim_{R \rightarrow +\infty} \int_{D^V(R)} f d\phi \left(\sum_{i=1}^{2n-1} \prod_{h \neq i} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \right) e^{-\sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} dh_{\alpha_1} \frac{1 - e^{-T^{\alpha_1} h_{\alpha_1}}}{h_{\alpha_1}} = 0.$$

Now we assume that there is one $\rho_{\beta_i} = 0$ near $x \in \text{Supp}(d\phi)$. Then by (2.53) and the Stokes formula, if $i \neq 2n$, we have near x that

$$\begin{aligned}
 (2.55) \quad & \int_{D^V(R)/M} \prod_{h \neq i} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) e^{-\sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} dh_{\alpha_1} \frac{1 - e^{-T^{\alpha_1} h_{\alpha_1}}}{h_{\alpha_1}} \\
 &= - \int_{\partial D^V(R)/M} \prod_{h \neq i} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) e^{-\sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} g(T^{\alpha_1} h_{\alpha_1}) = 0,
 \end{aligned}$$

as $h_{\beta_{2n}} = R$ is a constant on $\partial D^V(R)$.

By (2.54), (2.55) and a partition of unity argument, one gets

$$(2.56) \quad \lim_{T \rightarrow +\infty} \lim_{R \rightarrow +\infty} \int_{D^V(R)} d\phi \left(\sum_{i=1}^{2n-1} \prod_{h \neq i} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \right) e^{-\sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} dh_{\alpha_1} \frac{1 - e^{-T^{\alpha_1} h_{\alpha_1}}}{h_{\alpha_1}} = 0.$$

Lemma 2.8. *The following identity holds,*

$$(2.57) \quad \lim_{T \rightarrow +\infty} \lim_{R \rightarrow +\infty} \int_{D^V(R)} d\phi \prod_{h=1}^{2n-1} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) e^{-\sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} dh_{\alpha_1} \frac{1 - e^{-T^{\alpha_1} h_{\alpha_1}}}{h_{\alpha_1}} = 0.$$

Proof. By (2.53) and the Stokes formula, the integral in (2.57) equals to

$$(2.58) \quad - \int_{\text{Supp}(\phi)} \int_{\partial D^V(R)/M} d\phi \prod_{h=1}^{2n-1} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) e^{-\sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} g(T^{\alpha_1} h_{\alpha_1}) \\ + \int_{D^V(R)} d\phi \prod_{h=1}^{2n-1} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \rho_{\beta_{2n}} T^{\beta_{2n}} dh_{\beta_{2n}} e^{-\sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} g(T^{\alpha_1} h_{\alpha_1}).$$

If we write $d\phi = d(\phi - 1)$, then by the Stokes formula, the first term in (2.58) equals to

$$(2.59) \quad \int_{\partial(\text{Supp}(\phi))} \int_{\partial D^V(R)/M} \prod_{h=1}^{2n-1} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) e^{-\sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} g(T^{\alpha_1} h_{\alpha_1}) \\ - \int_{\text{Supp}(\phi)} \int_{\partial D^V(R)/M} (\phi - 1) \prod_{h=1}^{2n-1} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) d\rho_{\beta_{2n}} T^{\beta_{2n}} h_{\beta_{2n}} e^{-\sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} g(T^{\alpha_1} h_{\alpha_1}).$$

By (2.47) and (2.52), one has that

$$(2.60) \quad 0 \leq g(T^{\alpha_1} h_{\alpha_1}) \leq T^{\alpha_1} h_{\alpha_1}.$$

From (2.60), one sees that if one of $\rho_{\beta_i}(x) > 0$ for $x \in \partial(\text{Supp}(\phi))$, then there exists a sufficiently small open neighborhood W_x of $x \in \partial(\text{Supp}(\phi))$ such that for any $f \in C^\infty(\partial(\text{Supp}(\phi)))$ supported in W_x , one has

$$(2.61) \quad \lim_{R \rightarrow +\infty} \int_{\partial(\text{Supp}(\phi))} \int_{\partial D^V(R)/M} f \prod_{h=1}^{2n-1} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) e^{-\sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} g(T^{\alpha_1} h_{\alpha_1}) = 0.$$

If one of $\rho_{\beta_i} = 0$ near x , with $1 \leq i \leq 2n - 1$, then (2.61) holds trivially.

Thus we now deal with the case where $x \in (\cap_{i=1}^{2n-1} \partial U_{\beta_i}) \cap \partial(\text{Supp}(\phi))$ and $\rho_{\beta_{2n}} = 0$ near x . By transversality, such points are discrete on $\partial(\text{Supp}(\phi))$.

Let $f = 1$ near $x \in \partial(\text{Supp}(\phi))$ and supported close enough to x . Then as $g(0) = 0$, one has by the Stokes formula that

$$\begin{aligned}
(2.62) \quad & \int_{\partial(\text{Supp}(\phi))} \int_{\partial D^V(R)/M} f \prod_{h=1}^{2n-1} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) e^{-\sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} g(T^{\alpha_1} h_{\alpha_1}) \\
&= 2^{2n-1} \int_{\partial(\text{Supp}(\phi))} \int_{\partial D^V(R)/M} f \prod_{h=1}^{2n-1} \left(e^{(d^H - i_Y)(\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}/2)} \right) g(-(d^H - i_Y) T^{\alpha_1} dh_{\alpha_1}/2) \\
&= \int_{\partial(\text{Supp}(\phi))} \int_{\partial D^V(R)/M} df \left(\sum_{i=1}^{2n-1} \prod_{h \neq i} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \right) e^{-\sum_{i=1}^{2n-1} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} \frac{dh_{\alpha_1}}{h_{\alpha_1}} g(T^{\alpha_1} h_{\alpha_1}).
\end{aligned}$$

For any $x' \in \text{Supp}(df) \subset \partial(\text{Supp}(\phi))$, by the transversality, either one of ρ_{β_i} ($1 \leq i \leq 2n-1$) vanishes near x' , or one of them is nonzero at x' . In the later case, by (2.60) one finds that there exists a sufficiently small open neighborhood $W'_{x'}$ of $x' \in \partial(\text{Supp}(\phi))$ such that for any $\gamma \in C^\infty(\partial(\text{Supp}(\phi)))$ supported in $W'_{x'}$, one has that for any $1 \leq i \leq 2n-1$,

$$(2.63) \quad \lim_{R \rightarrow +\infty} \int_{\partial(\text{Supp}(\phi))} \int_{\partial D^V(R)/M} \gamma df \prod_{h \neq i} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) e^{-\sum_{i=1}^{2n-1} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} \frac{dh_{\alpha_1}}{h_{\alpha_1}} g(T^{\alpha_1} h_{\alpha_1}) = 0.$$

If one of $\rho_{\beta_j} = 0$, with $1 \leq j \leq 2n-1$, near x' , then (2.63) holds trivially for the product terms containing $d\rho_{\beta_j}$. Without loss of generality we assume $j = 1$ and deal with the following remaining term, by setting $\tilde{g}(x) = \int_0^x \frac{g(t)}{t} dt$,

$$\begin{aligned}
(2.64) \quad & \int_{\partial(\text{Supp}(\phi))} \int_{\partial D^V(R)/M} \gamma df \prod_{h=2}^{2n-1} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) e^{-\sum_{i=1}^{2n-1} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} \frac{dh_{\alpha_1}}{h_{\alpha_1}} g(T^{\alpha_1} h_{\alpha_1}) \\
&= \int_{\partial(\text{Supp}(\phi))} \int_{\partial D^V(R)/M} \gamma df d^V \left(\prod_{h=2}^{2n-1} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) e^{-\sum_{i=2}^{2n-1} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} \tilde{g}(T^{\alpha_1} h_{\alpha_1}) \right) = 0.
\end{aligned}$$

Thus (2.63) holds in general.

From (2.62), (2.63) and a partition of unity argument, one sees that (2.61) holds in the case where $x \in (\cap_{i=1}^{2n-1} \partial U_{\beta_i}) \cap \partial(\text{Supp}(\phi))$ and $\rho_{\beta_{2n}} = 0$ near x , if one assumes that $f = 1$ near x . Since there are only a finite number of such points, from (2.61) and a partition of unity argument, one gets the following estimate for the first term in (2.59),

$$(2.65) \quad \lim_{R \rightarrow +\infty} \int_{\partial(\text{Supp}(\phi))} \int_{\partial D^V(R)/M} \prod_{h=1}^{2n-1} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) e^{-\sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} g(T^{\alpha_1} h_{\alpha_1}) = 0.$$

For the second term in (2.59), for any $q \in \text{Supp}(\phi) \cap \text{Supp}(1 - \phi)$, clearly $q \neq p$ and then either one of $\rho_{\beta_i}(q) > 0$ ($1 \leq i \leq 2n$), or one of them vanishes near $q \in M$. In the

former case, by (2.60) one sees that there exists a sufficiently small open neighborhood V_q of $q \in M$ such that for any $f \in C^\infty(M)$ supported in V_q , one has

$$(2.66) \quad \lim_{R \rightarrow +\infty} \int_{\text{Supp}(\phi)} \int_{\partial D^V(R)/M} f(\phi - 1) \prod_{h=1}^{2n-1} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) d\rho_{\beta_{2n}} T^{\beta_{2n}} h_{\beta_{2n}} \\ \cdot e^{-\sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} g(T^{\alpha_1} h_{\alpha_1}) = 0,$$

while if one of ρ_{β_i} 's vanishes near q , (2.66) holds trivially.

From (2.59), (2.65), (2.66) and a partition of unity argument, one gets the required estimate for the first term in (2.58).

For the second term in (2.58), for any $q \in \text{Supp}(d\phi)$, either one of $\rho_{\beta_i}(q) > 0$ ($1 \leq i \leq 2n$), or one of them vanishes near q . By (2.60) and proceeding as in (2.35), one sees that there exists a sufficiently small open neighborhood V_q of $q \in M$ such that for any $f \in C^\infty(M)$ supported in V_q , one has

$$(2.67) \quad \lim_{T \rightarrow +\infty} \lim_{R \rightarrow +\infty} \int_{D^V(R)} f d\phi \prod_{h=1}^{2n-1} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \rho_{\beta_{2n}} T^{\beta_{2n}} dh_{\beta_{2n}} e^{-\sum_{i=1}^{2n} \rho_{\beta_i} T^{\beta_i} h_{\beta_i}} g(T^{\alpha_1} h_{\alpha_1}) = 0.$$

From (2.58), (2.59), (2.65)-(2.67) and a partition of unity argument, one gets (2.57), which completes the proof of Lemma 2.8. \square

Remark 2.9. The proof of Lemma 2.8 applies to give a proof of (2.56) without using the fact that $h_{\beta_{2n}}$ is constant on $\partial D^V(R)$.

From (2.51), (2.56) and (2.57), one gets

$$(2.68) \quad \lim_{T \rightarrow +\infty} \lim_{R \rightarrow +\infty} \int_{D^V(R)} \phi \prod_{h=1}^{2n} (d\rho_{\beta_h} T^{\beta_h} dh_{\beta_h}) \exp(-|Y|_T^2) = 0.$$

From Lemma 2.6 and (2.68), one gets (2.39).

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